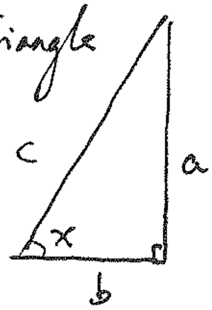
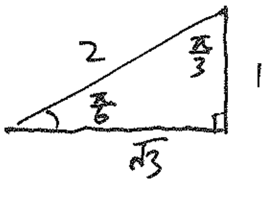
§6.6 Inverse Trigonometric Functions.

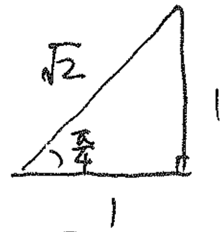
- **Right Triangle**



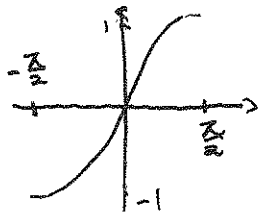
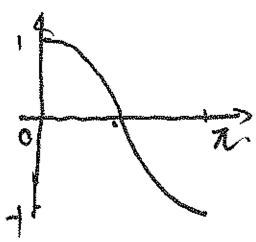
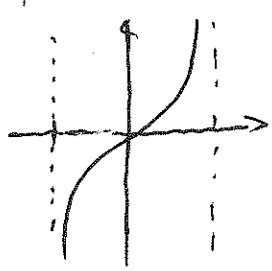
$$\boxed{\sin x = \frac{a}{c} \qquad \cos x = \frac{b}{c} \qquad \tan x = \frac{a}{b}}$$

$$\csc x = \frac{c}{a} \qquad \sec x = \frac{c}{b} \qquad \cot x = \frac{b}{a}.$$

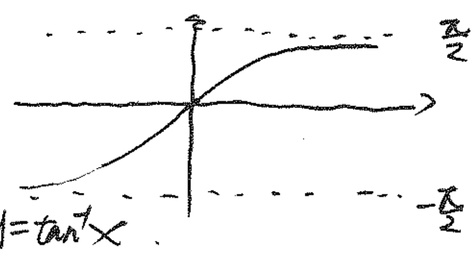




- **Inverse Trig**

	Graph	Inverse	Domain	Range
☆ $y=\sin x$, $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$.		$y=\sin^{-1}x$ (or $y=\arcsin x$)	$x\in[-1,1]$	$y\in[-\frac{\pi}{2},\frac{\pi}{2}]$
☆ $y=\cos x$, $x\in[0,\pi]$		$y=\cos^{-1}x$. $=\arccos x$	$x\in[-1,1]$	$y\in[0,\pi]$
☆ $y=\tan x$, $x\in(-\frac{\pi}{2},\frac{\pi}{2})$		$y=\tan^{-1}x$ $=\arctan x$	$x\in(-\infty,+\infty)$	$y\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$\boxed{\lim_{x\to\pm\infty}\tan^{-1}x = \pm\frac{\pi}{2}}$$



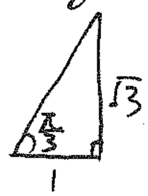
☆☆ $y=\sec x$. , $y=\sec^{-1}x$, $\begin{cases} x\geq 1, & y\in[0,\frac{\pi}{2}) \\ x\leq -1, & y\in[\pi,\frac{3\pi}{2}) \end{cases}$

$$\boxed{\begin{aligned} \lim_{x\to+\infty}\sec^{-1}x &= \frac{\pi}{2} \\ \lim_{x\to-\infty}\sec^{-1}x &= \frac{3\pi}{2}. \end{aligned}}$$

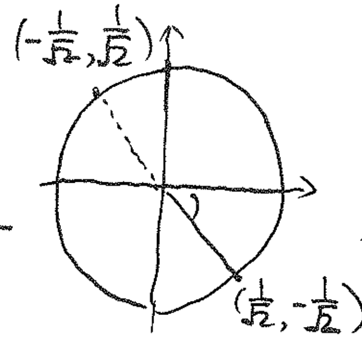
Rmk: $\sec^{-1}x$, $\csc^{-1}x$, $\cot^{-1}x$ are not ~~fre~~ used as frequently as others.

e.g 1 Solving Triangles.

$\tan^{-1}(\sqrt{3}) = \underline{\frac{\pi}{3}}$

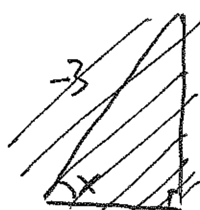


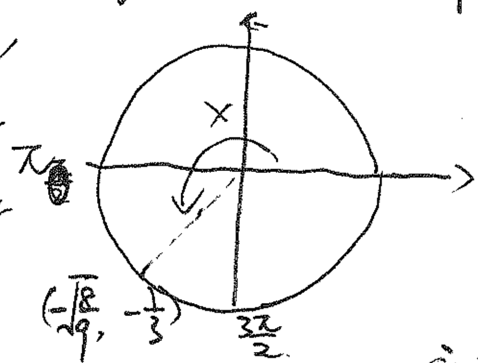
; $\tan^{-1}(-1) = \underline{-\frac{\pi}{4}}$



; $\cos^{-1}(-\frac{1}{\sqrt{2}}) = \underline{\frac{3}{4}\pi}$.

☆☆ e.g 2. suppose $\sec X = -3$, find $\sin X$, $\cot X$.
(Webwork) (It is equivalent to ask find $\sin(\sec^{-1}(-3))$, $\cot(\sec^{-1}(-3))$).





$\sec X = \frac{1}{-\frac{1}{3}}$ $\longrightarrow$ y coordinate in the unit disk

x coordinate: $\sqrt{1-(-\frac{1}{3})^2} = \sqrt{\frac{8}{9}}$.

$\sin X = -\sqrt{\frac{8}{9}} = \boxed{-\frac{2\sqrt{2}}{3}}$, $\cot X = \frac{-\sqrt{\frac{8}{9}}}{-\frac{1}{3}} = \boxed{2\sqrt{2}}$

- Derivatives: $\boxed{\frac{d}{dx}\tan^{-1}X = \frac{1}{1+X^2}, \quad \frac{d}{dx}\sin^{-1}X = \frac{1}{\sqrt{1-X^2}}}$

$(\cos^{-1}x)' = -\frac{1}{\sqrt{1-X^2}}$, $(\csc^{-1}x)' = -\frac{1}{x\sqrt{X^2-1}}$, $(\sec^{-1}x)' = \frac{1}{X\sqrt{X^2-1}}$, $(\cot^{-1}x)' = -\frac{1}{1+X^2}$.

Rmk: All these six formulas are given in the exam formula sheet. The first two should be memorized

e.g. 3. Find $f'(x)$ if $f(x) = \ln \cos^{-1}(\frac{X}{2})$

SLN: $f'(x) = \frac{1}{\cos^{-1}(\frac{X}{2})} \cdot \left(\cos^{-1}(\frac{X}{2})\right)' = \frac{1}{\cos^{-1}(\frac{X}{2})} \cdot -\frac{1}{\sqrt{1-(\frac{X}{2})^2}} \cdot (\frac{X}{2})' = \boxed{\frac{-1}{2\cos^{-1}(\frac{X}{2})\cdot\sqrt{1-\frac{X^2}{4}}}}$

e.g. 4. Evaluate $\int \frac{15}{\sqrt{25-X^2}}\,dX$

u-sub: $u = \frac{x}{5} \Leftrightarrow x = 5u$
$du = \frac{1}{5}dx \Leftrightarrow dx = 5du$

$= \int \frac{15}{\sqrt{25-25u^2}} \cdot 5du$

$= \int \frac{15}{\sqrt{25(1-u^2)}} \cdot 5du = \int \frac{15 \cdot \cancel{5}du}{\cancel{5}\sqrt{1-u^2}} = 15 \cdot \sin^{-1}u + C = 15 \cdot \sin^{-1}(\frac{X}{5}) + C$

☆ e.g. 5 Evaluate $\int \frac{x}{\sqrt{1-9x^4}}\,dx$. Idea: It might help to have $\sqrt{1-\square^2}$
(webwork)

s/n: u-sub: $u=3x^2$, $du=6x\cdot dx$.

$$\int \frac{x\,dx}{\sqrt{1-9x^4}} = \int \frac{1}{\sqrt{1-(3x^2)^2}}\cdot x\,dx = \int \frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{6}\,du = \frac{1}{6}\sin^{-1}u + C = \boxed{\frac{1}{6}\sin^{-1}(3x^2)+C}$$

☆ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$; $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$; $\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$

This one is not in the formula list

☆★ e.g. 6 $\int_2^4 \frac{1}{x^2-4x+8}\,dx$ Hint: Complete the square:

$$x^2-4x+8 = (x^2-4x+4)+4 = (x-2)^2+2^2$$

s/n: $u=x-2$, $du=dx$

$$= \int \frac{1}{(x-2)^2+2^2}\,dx$$

$$= \int \frac{1}{u^2+2^2}\,dx \xrightarrow[\text{with } a=2]{\text{apply the formula}} \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)\Big|_{x=2}^{x=4}$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{8}}.$$

☆☆ In general, the following formulas related to $\sin^{-1}$ and $\tan^{-1}$ can be derived via u-sub

$$\int \frac{dx}{\sqrt{1-b^2x^2}} = \frac{1}{b}\sin^{-1}(bx)+C,\quad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C,\quad \int \frac{dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+C$$

$$\int \frac{dx}{1+b^2x^2} = \frac{1}{b}\tan^{-1}(bx)+C,\quad \boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C},\quad \int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)+C$$

§6.7 Hyperbolic Functions

• Def: $\boxed{\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}}$, $\boxed{(\sinh x)' = \cosh x, \ (\cosh x)' = \sinh x}$

$\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$, $\operatorname{csch} x = \frac{1}{\sinh x}$, $\operatorname{sech} x = \frac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$ These are in the formula ~~sheet~~ (exam).

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eg.1 Simplify $5\sinh(2\ln 10) = 5 \cdot \frac{e^{2\ln 10} - e^{-2\ln 10}}{2} = 5 \cdot \frac{e^{\ln 10^2} - e^{\ln 10^{-2}}}{2}$
(WW)

$= 5 \cdot \frac{100 - \frac{1}{100}}{2} = \boxed{\frac{9999}{40}}$.

eg.2 Evaluate $\int \frac{\sinh \sqrt{x}}{\sqrt{x}} dx$. Hint: substitute the 'ugly' part
(S16)

$u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$= \int \sinh u \cdot 2du$

$= 2\cosh u + C = \boxed{2\cosh\sqrt{x} + C}$

eg.3. Compute y' where $y = \cosh(1+x^3)$. (chain rule)
(S15).

$\boxed{y' = \sinh(1+x^3) \cdot 3x^2}$

eg.4 Evaluate $\int \frac{\cosh(\ln(4x))}{x} dx$ Hint: Ugly part is $\ln(4x)$.
(F14).

$u = \ln 4x$, $du = \frac{1}{x} \cdot dx$.
(Caution: NOT $\frac{1}{4x}$)

$= \int \cosh(u) \cdot du$

$= \sinh u + C = \boxed{\sinh(\ln 4x) + C}$

eg.5. If $\sinh x = -5$, then $\tanh x = \underline{\frac{-5}{\sqrt{26}}}$
(WW).

$\cosh^2 x - \sinh^2 x = 1 \Rightarrow \cosh^2 x = 25 + 1 = 26 \Rightarrow \cosh x = \sqrt{26}$

eg 6 (ww) Find $f'(x)$ if $f(x)=e^{\tanh(3x)}$ based on $\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$

sln: $f'(x) = e^{\tanh(3x)} \cdot (\tanh(3x))' = \boxed{e^{\tanh(3x)} \cdot \mathrm{sech}^2(3x) \cdot 3}$

☆+ eg 7 (ww). Evaluate $\int \tanh(ax)\,dx$ (a is a constant)

Hint: recall how to evaluate $\int \tan x\,dx$.

$$\int \tanh(ax)\,dx = \int \frac{\sinh(ax)}{\cosh(ax)}\,dx$$

u-sub: $u=\cosh(ax)$
$du = \sinh(ax)\cdot a \cdot dx$
$\Rightarrow \frac{1}{a}du = \sinh(ax)\,dx$.

$$= \int \frac{1}{u}\cdot\frac{1}{a}\,du$$

$$= \frac{1}{a}\ln|u| + C = \boxed{\frac{1}{a}\ln|\cosh(ax)| + C}$$

eg 8 (ww). Evaluate $\int \mathrm{sech}^2(ax+b)\cdot dx$ Hint: $\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$.

$u = ax+b$, $du = a\cdot dx$

$$= \int \mathrm{sech}^2 u \cdot \frac{1}{a}\cdot du$$

$$= \frac{1}{a}\cdot \tanh u + C = \boxed{\frac{1}{a}\tanh(ax+b) + C}$$

Two Properties related to even/odd functions and their integral.

- f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$; f odd, $\int_{-a}^{a} f(x)\,dx = 0$.

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eg 9. Give $\sinh(-x) = -\sinh x$, prove that $f(x) = (\sinh(7x))^3$ is odd.

Proof: $f(-x) = (\sinh(-7x))^3 = (-\sinh(7x))^3 = -(\sinh(7x))^3$

$= -f(x)$. odd. #

§6.8. L'Hospital's Rule

- Basic limits: $\lim\limits_{x\to\pm\infty}\frac{1}{x}=\boxed{\frac{1}{\pm\infty}=0}$; $\lim\limits_{x\to 0^\pm}\frac{1}{x}=\boxed{\frac{1}{0^\pm}=\pm\infty}$

$\lim\limits_{x\to+\infty}e^x=\boxed{e^{+\infty}=+\infty}$; $\lim\limits_{x\to+\infty}\ln x=\boxed{\ln+\infty=+\infty}$; $\lim\limits_{x\to+\infty}\tan^{-1}x=\boxed{\tan^{-1}+\infty=\frac{\pi}{2}}$

$\lim\limits_{x\to-\infty}e^x=\boxed{e^{-\infty}=0}$; $\lim\limits_{x\to 0^+}\ln x=\boxed{\ln 0^+=-\infty}$; $\lim\limits_{x\to-\infty}\tan^{-1}x=\boxed{\tan^{-1}(-\infty)=-\frac{\pi}{2}}$

- Indeterminate Forms

e.g. 1 Find the limits: ① $\lim\limits_{x\to 0^+}\frac{\cos x}{2x}$ vs ② $\lim\limits_{x\to 0^+}\frac{\sin x}{2x}$.

① $=\frac{\cos 0^+}{2\cdot 0^+}=\frac{1}{0^+}=+\infty$ ② $=\frac{\sin 0^+}{2\cdot 0^+}=\boxed{\frac{0^+}{0^+}=?}$

Q: what is the meaning of "$\frac{0^+}{0^+}$"?

In general, the limits $\boxed{\frac{0}{0}}$ and $\boxed{\frac{\infty}{\infty}}$ are called INDETERMINATE forms.

Rmk: There are several other harder types: $\boxed{0\cdot\infty}$, $\infty-\infty$, 0^0, 1^∞, ∞^0

- L'Hospital Rule (L.H.)

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate type, then $\boxed{\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}}$.

② : $\lim\limits_{x\to 0^+}\frac{\sin x}{2x}\overset{L'H.}{=\!=}\lim\limits_{x\to 0^+}\frac{(\sin x)'}{(2x)'}=\lim\limits_{x\to 0^+}\frac{\cos x}{2}=\frac{\cos 0^+}{2}=\boxed{\frac{1}{2}}$.

- $\frac{0}{0}$ and $\frac{\infty}{\infty}$ types:

e.g. 2. Evaluate the following limits:

① $\lim\limits_{x\to\infty}\frac{\ln x}{\ln 2x}\overset{L'H}{=}\lim\limits_{x\to\infty}\frac{(\ln x)'}{(\ln 2x)'}=\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2x}\cdot 2}=\boxed{1}$

② $\lim_{x\to 0} \frac{e^{2x}-1}{x^2+3x} \overset{L'Hop}{=\!=} \lim_{x\to 0} \frac{e^{2x}\cdot 2}{2x+3} \overset{x=0}{=\!=} \frac{e^0\cdot 2}{2\cdot 0+3} = \boxed{\frac{2}{3}}$

• $0\cdot\infty$ Type : Idea: $\frac{1}{0}=\infty$, $\frac{1}{\infty}=0$. $0\cdot\infty$ can be rewritten as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ type.

eg. 3 (s16) $\lim_{x\to 0^+} x\cdot\ln(2x) = \lim_{x\to 0^+} \frac{\ln(2x)}{\frac{1}{x}}$ $\quad \frac{\ln 0 = -\infty}{\frac{1}{0} = \infty}$ type.

($0\cdot\infty$ type) $\overset{L'Hop}{=\!=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$ $\qquad$ Hint: $(x^n)' = n\cdot x^{n-1}$

$(\frac{1}{x})' = (x^{-1})' = -1\cdot x^{-2}$

$\overset{simplify}{=\!=} \lim_{x\to 0^+} -x = \boxed{0}$

☆☆. eg. 4. More harder examples:

0^0) ① (s15). $\lim_{x\to 0^+} (2x)^{3x}$ $\quad$ Hint: use identity $(2x)^{3x} = e^{\ln(2x)^{3x}}$, evaluate $\lim_{x\to 0^+} \ln(2x)^{3x}$

s/n: $\lim_{x\to 0^+} \ln(2x)^{3x} = \lim_{x\to 0^+} 3\cdot \underbrace{x\cdot\ln(2x)}_{eg.3} = 3\cdot 0 = 0$

Then $\lim_{x\to 0^+} (2x)^{3x} = \lim_{x\to 0^+} e^{\ln(2x)^{3x}} = e^{\lim_{x\to 0^+} \ln(2x)^{3x}} = e^0 = \boxed{1}$.

$\infty-\infty$) ② $\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1} \frac{x-1-\ln x}{\ln x\cdot(x-1)}$ $\qquad \frac{0}{0}$ type

0

$\overset{L'Hop}{=\!=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x\cdot 1 + \frac{1}{x}\cdot(x-1)} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}}$ $\quad$ (still $\frac{0}{0}$ type) need L'Hop one more time

$\overset{L'Hop}{=\!=} \lim_{x\to 1} \frac{-(-\frac{1}{x^2})}{\frac{1}{x} + \frac{1}{x^2}}$

$= \frac{1}{1+1} = \boxed{\frac{1}{2}}$

eg. 5 Important Formula for e. $e = \lim_{x\to+\infty} (1+\frac{1}{x})^x$, $e^a = \lim_{x\to+\infty} (1+\frac{a}{x})^x$

$\lim_{x\to\infty} \ln(1+\frac{a}{x})^x = \lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{\frac{1}{x}} \overset{L'Hop}{=\!=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\cdot(-\frac{a}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} = a$

Then $\lim_{x\to\infty} (1+\frac{a}{x})^x = \boxed{e^a}$